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Cartan–Leray spectral sequence for Galois coverings of linear categories

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Abstract

We provide a Cartan–Leray type spectral sequence for the Hochschild–Mitchell (co)homology of a Galois covering of linear categories. We infer results relating the Galois group and Hochschild cohomology in degree one.

Keywords: Cartan; Leray; Covering; Hochschild; Mitchell; Category; Homology; Cohomology

1. Introduction

This past years several results and tools from algebraic topology has been adapted to representation theory of non commutative associative finite dimensional algebras over a field k. For this purpose k-categories has been considered as Galois coverings of algebras by P. Gabriel, see for instance [9] or also [3]. Recall that a k-category is a small category where morphisms are k-vector spaces and composition is k-bilinear. Relations between the representation theory of a k-algebra and functors from the universal cover has also been

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described and the fundamental group of the presentation of an algebra by a quiver with relations emerged in this context, see, e.g. [9].

Moreover, a precise relation which can be compared with Hurewicz's Theorem (see for instance [16]), has been obtained in [1,19] between the fundamental group of a presentation and Hochschild cohomology of the algebra in degree one. More recently, an interpretation of this algebraic fundamental group as a fundamental group of a simplicial complex or of a classifying CW-complex has been described, see [4,5,21]. Note also that comparisons between Hochschild cohomologies of a Galois covering are obtained in [14]. In [18,20] the relation between skew group algebras and coverings is studied and results are provided for a finite Galois group in semi-simple characteristic.

Our main purpose in this paper is to provide a setting relating the Galois group of the covering of a k-algebra (or more generally of a k-category) with the Hochschild (co)homology theory of the situation. A model for this is the Cartan–Leray spectral sequence associated to a group acting freely and properly on a connected space, see [6,7] and for instance [16, p. 337] or [22, p. 206].

We provide a Cartan–Leray type spectral sequence for Galois coverings of k-categories. For this purpose we briefly recall the so-called Hochschild–Mitchell homology and cohomology theories $H_*(\mathcal{C}, M)$ and $H^*(\mathcal{C}, M)$ of a k-category $\mathcal{C}$ with coefficients in a bimodule $\mathcal{M}$ over $\mathcal{C}$. These (co)homology theories have been introduced by Mitchell in [17], and are closely related with theories considered in [11,15] but differs from Quillen's homology described in [12, Appendix C]. Notice that the spectral sequence results we obtain are also valid in case of a commutative ring k instead of a field, provided that we consider k-categories with flat module morphisms. Hochschild–Mitchell (co)homology theories are used in theoretical informatics and we thank speakers and participants of a seminar in Montpellier related to the subject for pointing out this theory.

In case of a k-category with a finite number of objects, the Hochschild–Mitchell (co)homology coincides with the usual Hochschild (co)homology of the k-algebra associated to the category, namely the direct sum of the morphisms spaces equipped with the matrix product. We provide a proof of this well-known agreement property, which in turn enables us to prove that Hochschild–Mitchell theory is invariant under contraction and expansion of a k-category—precise definitions of these operations are provided in the text.

A difference with the Cartan–Leray spectral sequence in the algebraic topology context is the use of coefficients. For Hochschild–Mitchell (co)homology, coefficients are taken in a bimodule which has to be lifted through the canonical projection, while in the topological context an unchanged abelian group of coefficients is used.

A cohomological type spectral sequence cannot be obtained in general for an infinite category with a group acting freely. Indeed Hochschild–Mitchell cohomology makes use of products of infinitely many k-nerves of the category and we note that this cannot be avoided. The action of the group on those cochains do not provide free modules over the group algebra, hence the spectral sequence does not necessarily collapse for the column filtration.

Nevertheless our interest is mainly in Hochschild cohomology of a finite dimensional algebra with coefficients in itself. An usual duality holds between Hochschild–Mitchell homology and cohomology for locally finite bimodules, that is, bimodules $\mathcal{M}$ with values in the category of finite dimensional vector spaces. The dual $D\mathcal{M}$ of any bimodule is well defined and in case $\mathcal{M}$ is locally finite we have $DD\mathcal{M} = \mathcal{M}$.

This duality provides a Cartan–Leray type spectral sequence of cohomological type with coefficients in a locally finite bimodule. As a particular case of interest we consider a finite dimensional k-algebra Λ such that the larger semi-simple quotient is a product of copies of k, provided with a complete system of primitive orthogonal idempotents. Let $\mathcal{B}$ be the corresponding k-category and let $\mathcal{C} \to \mathcal{B}$ be a Galois covering with group G. Then the spectral sequence provides a canonical embedding $\operatorname{Hom}(G, k^+)$ in $H^1(\mathcal{B}, \mathcal{B})$. Note that this result has been obtained by Assem and de la Peña in [1] in case G is the fundamental group of a presentation of $\mathcal{B}$ by a triangular quiver with admissible relations, afterwards de la Peña and Saorín noticed in [19] that the triangular assumption is not needed.

As a consequence of our result, the dimension of $H^1(\mathcal{B}, \mathcal{B})$ is bounded below by the rank of G, compare with [14] where this inequality is obtained in case of a free group G acting freely on a schurian category $\mathcal{C}$.

In [14] a comparison between the Hochschild cohomology of the algebras involved in a Galois covering $A \to B$ with a cyclic group of order 2 is initiated, where B is the Kronecker algebra. Our considerations show that for a field of characteristic not 2 the 3-dimensional vector space $H^1(B)$ is actually isomorphic to the fixed points of the 5-dimensional vector space $H^1(A, LB)$, where LB is the lifted bimodule.

More generally let $\mathcal{C} \to \mathcal{B}$ be a Galois covering of finite categories with finite group G, and assume that the characteristic of k is zero or do not divide the order of G. We infer from the Cartan–Leray spectral sequence that

$$H^n(\mathcal{B}, \mathcal{M}) = H^n(\mathcal{C}, L\mathcal{M})^G$$

where $L\mathcal{M}$ is the lifting of the $\mathcal{B}$-bimodule $\mathcal{M}$ to the category $\mathcal{C}$.

2. Hochschild–Mitchell (co)homology

A category is called *small* if the objects and morphisms are sets, all categories in this paper are small. Let k be a field, a *k-category* is a small category such that the morphisms are k-vector spaces and the composition of morphisms is k-bilinear.

Let $\mathcal{C}$ be a k-category with objects set $\mathcal{C}_0$. It is convenient to denote by ${}_y\mathcal{C}_x$ the morphisms from the object x to the object y, composition provides k-linear maps

$${}_z\mathcal{C}_y \otimes_k {}_y\mathcal{C}_x \to {}_z\mathcal{C}_x$$

for $x, y, z \in \mathcal{C}_0$.

A *bimodule* $\mathcal{M}$ over a k-category $\mathcal{C}$ is a bifunctor from $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ to the category of vector spaces. In other words, $\mathcal{M}$ is given by a set of vector spaces $\{{}_y\mathcal{M}_x\}_{x,y\in\mathcal{C}_0}$ and left and right actions

$${}_z\mathcal{C}_y \otimes {}_y\mathcal{M}_x \to {}_z\mathcal{M}_x, \qquad {}_y\mathcal{M}_x \otimes {}_x\mathcal{C}_u \to {}_y\mathcal{M}_u$$

satisfying the usual associativity conditions, namely:

$$ {}_u c_z ({}_z c_y \, {}_y m_x) = ({}_u c_z \, {}_z c_y) {}_y m_x, \qquad {}_y m_x ({}_x c_u \, {}_u c_v) = ({}_y m_x \, {}_x c_u) {}_u c_v, $$
$$ ({}_z c_y \, {}_y m_x) {}_x c_u = {}_z c_y \, ({}_y m_x {}_x c_u), \qquad {}_y 1_y \, {}_y m_x = {}_y m_x \, {}_x 1_x = {}_y m_x. $$

For instance, the standard bimodule over a k-category is the category itself.

Of course, if $\mathcal{C}$ is finite the vector space

$$ \Lambda_{\mathcal{C}} = \bigoplus_{x,y \in \mathcal{C}_0} {}_y\mathcal{C}_x $$

has an algebra structure, with a well defined matrix product given by the composition of $\mathcal{C}$, the identity is the matrix with ${}_x 1_x$ in the diagonal and zero otherwise. Note that $\Lambda_{\mathcal{C}}$ has by construction a complete set of orthogonal idempotents, namely $\{{}_x 1_x\}_{x \in \mathcal{C}}$. Usual $\Lambda_{\mathcal{C}}$-bimodules and $\mathcal{C}$-bimodules coincide.

Conversely, let Λ be a k-algebra equipped with a complete set F of orthogonal idempotents (not necessarily primitive), that is, a finite subset F of Λ whose elements satisfy $x^2 = x$, $xy = 0$ if $x \neq y$ and $\sum_{x \in F} x = 1$. The associated finite category $\mathcal{C}_{\Lambda,F}$ has set of objects F and $y\Lambda x$ are the morphisms from x to y. Composition is given by the product in Λ.

Definition 2.1. Let $(x_{n+1}, x_n, \ldots, x_1)$ be a $(n+1)$-sequence of objects of a k-category $\mathcal{C}$. The k-nerve associated to this sequence is the vector space

$$ {}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_2}\mathcal{C}_{x_1}. $$

The k-nerve $\mathcal{N}_n$ of degree n is the direct sum of all the k-nerves associated to $(n+1)$-sequences of objects. We have

$$ \mathcal{N}_0 = \bigoplus_{x \in \mathcal{C}_0} {}_x\mathcal{C}_x, \qquad \mathcal{N}_n = \bigoplus_{(n+1)\text{-sequences}} {}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_2}\mathcal{C}_{x_1}. $$

Definition 2.2. Let $\mathcal{C}$ be a k-category and let $\mathcal{M}$ be a $\mathcal{C}$-bimodule. The Hochschild–Mitchell cohomology $H^*(\mathcal{C}, \mathcal{M})$ is the cohomology of the cochain complex

$$ 0 \to \prod_x {}_x\mathcal{M}_x \xrightarrow{d} \mathrm{Hom}(\mathcal{N}_1, \mathcal{M}) \xrightarrow{d} \cdots \xrightarrow{d} \mathrm{Hom}(\mathcal{N}_n, \mathcal{M}) \xrightarrow{d} \cdots $$

where

$$ C^n(\mathcal{C}, \mathcal{M}) = \mathrm{Hom}(\mathcal{N}_n, \mathcal{M}) = \prod_{(n+1)\text{-sequences}} \mathrm{Hom}_k\left({}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_2}\mathcal{C}_{x_1}, {}_{x_{n+1}}\mathcal{M}_{x_1}\right) $$

and d is given by the usual formulas of Hochschild cohomology as follows. Let $f = \{f_{(x_{n+1},\ldots,x_1)}\}_{(n+1)\text{-sequences}}$ be a family of linear maps. Then df is the family of linear maps

$$
\begin{aligned}
&(df) = \{df_{(x_{n+2},\dots,x_1)}\}_{(n+2)\text{-sequences}} \quad \text{where} \\
&df_{(x_{n+2},\dots,x_1)}({}_{x_{n+2}}c_{x_{n+1}} \otimes \cdots \otimes {}_{x_2}c_{x_1}) \\
&\quad = (-1)^{n+1}({}_{x_{n+2}}c_{x_{n+1}}) f_{(x_{n+1},\dots,x_1)}({}_{x_{n+1}}c_{x_n} \otimes \cdots \otimes {}_{x_2}c_{x_1}) \\
&\qquad + \sum_{i=1}^{i=n} (-1)^i f_{(x_{n+2},\dots,x_{i+2},x_i,\dots,x_1)}({}_{x_{n+2}}c_{x_n} \otimes \cdots \otimes ({}_{x_{i+2}}c_{x_{i+1}})({}_{x_{i+1}}c_{x_i}) \otimes \cdots \otimes {}_{x_2}c_{x_1}) \\
&\qquad + f_{(x_{n+2},\dots,x_2)}({}_{x_{n+2}}c_{x_{n+1}} \otimes \cdots \otimes {}_{x_3}c_{x_2})({}_{x_2}c_{x_1}).
\end{aligned}
$$

Note that each linear map of the family df is well defined.

Remark 2.3. The tensor product of $\mathcal{C}$-bimodules $\mathcal{M}_1$ and $\mathcal{M}_2$ is the bimodule given by

$$
{}_y(\mathcal{M}_1 \otimes \mathcal{M}_2)_x = \bigoplus_z {}_y(\mathcal{M}_1)_z \otimes {}_z(\mathcal{M}_2)_x.
$$

As in the algebra case, the standard bimodule $\mathcal{C}$ has a resolution by tensor powers of $\mathcal{C}$ which are projective bimodules. Applying the functor $\operatorname{Hom}(-, \mathcal{M})$ provides precisely the cochain complex above. Consequently Hochschild–Mitchell cohomology is an instance of an Ext functor. In particular

$$
H^0(\mathcal{C}, \mathcal{M}) = \left\{({}_x m_x)_x \mid {}_x m_x \in {}_x\mathcal{M}_x \text{ and } {}_y f_x\, {}_x m_x = {}_y m_y\, {}_y f_x \text{ for all } {}_y f_x \in {}_y\mathcal{C}_x\right\}.
$$

Hence $H^0(\mathcal{C}, \mathcal{C})$ is the center of the category.

Remark 2.4. Replacing the direct product by a direct sum in the cochain complex has no sense since the Hochschild coboundary will provide in general a direct product of morphisms, even as an image of a morphism located in a single component. However if the category is locally finite the direct sums of morphisms at the cochain level provide a subcomplex.

Definition 2.5. Let $\mathcal{C}$ be a k-category and let $\mathcal{M}$ be a $\mathcal{C}$-bimodule. The Hochschild–Mitchell homology $H_*(\mathcal{C}, \mathcal{M})$ is the homology of the chain complex

$$
\cdots \to \mathcal{M} \otimes \mathcal{N}_n \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{M} \otimes \mathcal{N}_1 \to \bigoplus_x {}_x\mathcal{M}_x \to 0
$$

where

$$
C_n(\mathcal{C}, \mathcal{M}) = \mathcal{M} \otimes \mathcal{N}_n = \bigoplus_{(n+1)\text{-sequences}} {}_{x_1}\mathcal{M}_{x_{n+1}} \otimes {}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_2}\mathcal{C}_{x_1}
$$

and d is given by the usual Hochschild boundary, see, for instance, [12].

Again this is $\operatorname{Tor}_*(\mathcal{C}, \mathcal{M})$ in the abelian category of $\mathcal{C}$-bimodules.

Remark 2.6. Hochschild–Mitchell homology and cohomology theories are not homology or cohomology of small categories as considered in [12, Appendix C]. The following well-known agreement result between usual Hochschild and Hochschild–Mitchell (co)homology theory holds.

Proposition 2.7. *Let $\mathcal{C}$ be a finite k-category, $\Lambda_{\mathcal{C}}$ the corresponding algebra and $\mathcal{M}$ a $\Lambda_{\mathcal{C}}$-bimodule. Then*

$$H^*(\mathcal{C},\mathcal{M}) = H^*(\Lambda_{\mathcal{C}},\mathcal{M}) \quad \textit{and} \quad H_*(\mathcal{C},\mathcal{M}) = H_*(\Lambda_{\mathcal{C}},\mathcal{M})$$

where $H(\Lambda_{\mathcal{C}},\mathcal{M})$ denotes usual Hochschild (co)homology of the algebra $\Lambda_{\mathcal{C}}$.

Proof. Let $\Lambda := \Lambda_{\mathcal{C}}$ and consider the semi-simple subalgebra of Λ given by $E = \prod k_x 1_x$. Note that any E-bimodule is projective since the enveloping algebra of E is still semi-simple. Hence $\Lambda \otimes_E X \otimes_E \Lambda$ is a projective Λ-bimodule for any E-bimodule X. Consequently there is a projective resolution of Λ as a Λ-bimodule given by

$$\cdots \to \Lambda \otimes_E \Lambda \otimes_E \cdots \otimes_E \Lambda \to \cdots \to \Lambda \otimes_E \Lambda \to \Lambda \to 0.$$

Applying the functor $\mathrm{Hom}_{\Lambda-\Lambda}(-,\mathcal{M})$ to this resolution and considering the canonical vector-space isomorphism

$$\mathrm{Hom}_{\Lambda-\Lambda}(\Lambda \otimes_E X \otimes_E \Lambda, \mathcal{M}) = \mathrm{Hom}_{E-E}(X,\mathcal{M})$$

we obtain a cochain complex computing $H^*(\Lambda,\mathcal{M})$ which coincides with the complex we have defined for the Hochschild–Mitchell cohomology. The same argument shows the assertion for homology. □

This result has an immediate generalization that we provide although we will not make direct use of it, however it can serve as a guideline for the reader. Notice first that there are two operations that we can perform on a k-category.

Let $\mathcal{F}$ be a finite full subcategory of a k-category $\mathcal{C}$. The *contraction* of $\mathcal{C}$ along $\mathcal{F}$ is the category $\mathcal{C}_{\mathcal{F}}$ where $\mathcal{F}$ is replaced with one object having $\Lambda_{\mathcal{F}}$ as endomorphism algebra and the evident morphisms between this new object and the rest of the objects of $\mathcal{C}$.

The *expansion* of $\mathcal{C}$ along a complete system F of orthogonal idempotents of ${}_x\mathcal{C}_x$ for an object x is given by the category obtained by replacing the object x by the set F and providing morphisms in accordance.

Of course a bimodule $\mathcal{M}$ over $\mathcal{C}$ provides a uniquely determined bimodule over a contraction or an expansion of $\mathcal{C}$ and clearly we have the following result.

Proposition 2.8. *The Hochschild–Mitchell (co)homology is invariant under contraction and expansion.*

Finally we note that the cyclic Mitchell homology of a category is also well defined along this lines.

3. Cartan–Leray spectral sequence

Let $\mathcal{C}$ be a k-category and G be a group acting freely on it, namely there is an action on objects such that for any $s \in G$, $x \in \mathcal{C}_0$, $sx = x$ implies $s = 1$, and an action on the morphisms such that for any $c \in {}_y\mathcal{C}_x$ we have $sc \in {}_{sy}\mathcal{C}_{sx}$ and $s(dc) = sdsc$ for any $d \in {}_z\mathcal{C}_y$, $c \in {}_y\mathcal{C}_x$. Of course $s(tx) = (st)x$ and $s(tc) = (st)c$ for any $s, t \in G$, $x \in \mathcal{C}_0$ and c a morphism in $\mathcal{C}$.

The quotient category $\mathcal{C}/G$ has objects the set of orbits of objects. The morphisms between two orbits is the direct sum of the orbit spaces of morphisms, namely let u, v be orbits of objects, then $(\bigoplus_{x\in u, y\in v} {}_y\mathcal{C}_x)$ is a left kG-module and we put

$$ {}_v(\mathcal{C}/G)_u = \left(\bigoplus_{x\in u,\ y\in v} {}_y\mathcal{C}_x\right)\Big/G $$

where as usual we denote X/G the quotient of a kG-module X by the action of G, namely $X/G = X/(\operatorname{Ker}\varepsilon)X$ where $\varepsilon : kG \to k$ is the augmentation algebra morphism defined by $\varepsilon(s) = 1$ for all $s \in G$. In particular $kG/G = k$.

Composition of morphisms is well defined precisely because the G-action on objects of $\mathcal{C}$ is free.

Remark 3.1. P. Gabriel has considered in [9, p. 85] a group G acting freely on a k-category and fixed points on families of morphisms. The fixed points approach impose restrictions on the categories considered in [9] which we do not need in our context. In the context of boxes, Y.A. Drozd and S.A. Ovsienko have introduced in [8] the same categorical quotient that we consider in this paper, we thank L. Salmerón for pointing out this.

Remark 3.2. Let x_0 be a fixed object in an orbit u, in other words $u = \overline{x_0}$ where $\overline{x_0}$ denotes the G-orbit of an object of $\mathcal{C}$. Then

$$ {}_v(\mathcal{C}/G)_u = \bigoplus_{y\in v} {}_y\mathcal{C}_{x_0}. $$

Definition 3.3. A Galois covering of k-categories under the action of a group G is a functor $\mathcal{C} \to \mathcal{B}$ where G acts freely on $\mathcal{C}$ and $\mathcal{B} = \mathcal{C}/G$. The functor is the projection functor.

Definition 3.4. Let $\mathcal{M}$ be a $\mathcal{B}$-bimodule. The lifted $\mathcal{C}$-bimodule $L\mathcal{M}$ is defined by

$$ {}_y(L\mathcal{M})_x = {}_{\bar{y}}\mathcal{M}_{\bar{x}} $$

with left action

$$ {}_z\mathcal{C}_y \otimes {}_y(L\mathcal{M})_x \to {}_z(L\mathcal{M})_x $$

given by ${}_zc_y\, {}_ym_x = {}_z({}_{\bar{z}}\bar{c}_{\bar{y}}\, {}_{\bar{y}}m_{\bar{x}})_x$, and right action defined analogously.

Note that this definition is in force for any functor between categories and, in particular, for a Galois covering. However for a Galois covering the group G acts on a $\mathcal{C}$-bimodule $\mathcal{U}$: the $\mathcal{C}$-bimodule ${}^s\mathcal{U}$ is given by the vector spaces ${}_y({}^s\mathcal{U})_x = {}_{s^{-1}y}\mathcal{U}_{s^{-1}x}$ and left action

$$c.u = \left(s^{-1}c\right)(u) \in {}_{s^{-1}z}\mathcal{U}_{s^{-1}x}, \quad c \in {}_z\mathcal{C}_y,$$

and similarly on the right. Clearly a lifted bimodule is fixed under this action.

Proposition 3.5. *Let $\mathcal{U}$ be a $\mathcal{C}$-bimodule. Then for each $s \in G$ there is an induced map between Hochschild–Mitchell homologies*

$$H_*(\mathcal{C}, \mathcal{U}) \to H_*\left(\mathcal{C}, {}^s\mathcal{U}\right).$$

Proof. Consider at the Hochschild–Mitchell chain level the map given as follows, (tensor signs are replaced by commas)

$$\left({}_{x_1}(u)_{x_{n+1}}, {}_{x_{n+1}}(c_n)_{x_n}, \dots, {}_{x_2}(c_1)_{x_1}\right) \mapsto (u, sc_n, \dots, sc_1),$$

where u on the right hand side belongs to ${}_{sx_1}({}^s\mathcal{U})_{sx_{n+1}} = {}_{x_1}(\mathcal{U})_{x_{n+1}}$. This is clearly a chain map as a direct consequence of the definitions of left end right actions on ${}^s\mathcal{U}$. $\square$

Corollary 3.6. *Let $\mathcal{M}$ be a $\mathcal{B}$-bimodule and $L\mathcal{M}$ the corresponding lifted $\mathcal{C}$-bimodule. Then the Galois group G of the covering acts on $H_*(\mathcal{C}, L\mathcal{M})$.*

Proof. As already quoted we have ${}^s(L\mathcal{M}) = L\mathcal{M}$ for any $s \in G$. $\square$

Consequently $H_n(\mathcal{C}, L\mathcal{M})$ is a kG-module for each $n \geqslant 0$.

Proposition 3.7. *Let $\mathcal{C} \to \mathcal{B} = \mathcal{C}/G$ be a Galois covering of k-categories with group G and let $\mathcal{M}$ be a $\mathcal{B}$-bimodule. Then $C_*(\mathcal{C}, L\mathcal{M})$ is a free kG-module and*

$$C_*(\mathcal{C}, L\mathcal{M})/G = C_*(\mathcal{B}, \mathcal{M}).$$

Proof. The k-nerve of degree n of $\mathcal{C}$ is a free kG-module which decomposes as a direct sum of kG-modules along G-orbits of $n+1$ objects, each of the summands is a free kG-module. The Hochschild–Mitchell chain complex corresponding to the orbit of a given $(n+1)$-sequence $x_{n+1}, \dots, x_1$ is obtained by tensoring the summand of the nerve with the constant vector space ${}_{\bar{x}_1}\mathcal{M}_{\bar{x}_{n+1}}$, providing in this way that each kG-direct summand of $C_*(\mathcal{C}, L\mathcal{M})$ is kG-free.

Let now $u_{n+1}, \dots, u_1$ be a sequence of G-orbits of objects of $\mathcal{C}$. We assert that the two following vector spaces are isomorphic:

$$A = \left[\bigoplus_{x_i \in u_i} \left({}_{x_1}(L\mathcal{M})_{x_{n+1}} \otimes {}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_3}\mathcal{C}_{x_2} \otimes {}_{x_2}\mathcal{C}_{x_1}\right)\right]/G$$

and

$$B = {}_{u_1}\mathcal{M}_{u_{n+1}} \otimes {}_{u_{n+1}}(\mathcal{C}/G)_{u_n} \otimes \cdots \otimes {}_{u_3}(\mathcal{C}/G)_{u_2} \otimes {}_{u_2}(\mathcal{C}/G)_{u_1}.$$

Let $(m, c_n, \ldots, c_1)$ be a representative of A, then

$$\varphi(m, c_n, \ldots, c_1) = (m, \bar{c}_n, \ldots, \bar{c}_1)$$

provides a well defined element of B. Conversely let $(m, \gamma_n, \ldots, \gamma_2, \gamma_1) \in B$ and we choose any $c_1 \in \gamma_1$. We adjust the choice of $c_2 \in \gamma_2$ in such a way that the source object of c_2 is the target one of c_1, this can be done in an unique way since the action of G on the category is free. We pursue the right choices until $c_n \in \gamma_n$. Concerning $m \in {}_{u_1}\mathcal{M}_{u_{n+1}}$, choose to consider it in ${}_{x_1}(L\mathcal{M})_{x_{n+1}}$ and put

$$\psi(m, \gamma_n, \ldots, \gamma_1) = (m, c_n, \ldots, c_1) \in \left[{}_{x_1}(L\mathcal{M})_{x_{n+1}} \otimes {}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_2}\mathcal{C}_{x_1}\right]/G.$$

The point is to prove that this element does not depend on the choice we made of a representative c_1 of the class γ_1. Let c_1' be another choice. There exist a unique $s \in G$ such that $c_1' = sc_1$. Clearly the unique c_2' such that its source is the target of c_1' is given by $c_2' = sc_2$, the same group element s is used. Finally, $c_n' = sc_n$ and m is now located in ${}_{sx_1}(L\mathcal{M})_{sx_{n+1}}$. Note that

$$(m, sc_n, \ldots, sc_1) = s(m, c_n, \ldots, c_1),$$

so the elements coincide in A. □

Remark 3.8. The proof above shows that it is crucial for the bimodule $L\mathcal{M}$ to verify

$${}_{x_1}(L\mathcal{M})_{x_{n+1}} = {}_{sx_1}(L\mathcal{M})_{sx_{n+1}} = {}_{\bar{x}_1}\mathcal{M}_{\bar{x}_{n+1}}.$$

Remark 3.9. The dual statement for cohomology does not hold in general for an infinite group. Namely Hochschild–Mitchell cochain $C^n(\mathcal{C}, L\mathcal{M})$ is an infinite product of free kG-modules which is not a free module in general.

Theorem 3.10 (Cartan–Leray). *Let $\mathcal{C} \to \mathcal{B}$ be a Galois covering of k-categories with group G and let $\mathcal{M}$ be a $\mathcal{B}$-bimodule. There is a first quadrant spectral sequence converging to $H_n(\mathcal{B}, \mathcal{M})$ with level* 2 *term*

$$E^2_{p,q} = H_p\big(G, H_q(\mathcal{C}, L\mathcal{M})\big)$$

where $H_p(G, X)$ denotes usual group homology with coefficients in a kG-module X.

We will prove this Theorem following the classical proof in algebraic topology. Before, we consider a Cartan–Leray Theorem for Hochschild–Mitchell cohomology which is of special interest for finite dimensional k-algebras.

Theorem 3.11 (Cartan–Leray). *Let $\mathcal{C} \to \mathcal{B}$ be a Galois covering of k-categories with group G and let $\mathcal{M}$ be a locally finite $\mathcal{B}$-bimodule. There is a first quadrant spectral sequence of cohomological type converging to $H^n(\mathcal{B}, \mathcal{M})$ with level 2 term*

$$E_2^{p,q} = H^p\big(G, H^q(\mathcal{C}, L\mathcal{M})\big)$$

where $H^p(G, X)$ denotes usual group cohomology with coefficients in a kG-module X.

Proof. For any $\mathcal{B}$-bimodule $\mathcal{Y}$ we have

$$H_n(\mathcal{B}, Y)^* = H^n(\mathcal{B}, D\mathcal{Y})$$

where * denotes the dual vector space and $D\mathcal{Y}$ is the dual bimodule of $\mathcal{Y}$. Recall that a locally finite bimodule $\mathcal{M}$ has finite dimensional vector spaces ${}_y\mathcal{M}_x$ for each couple of objects (y, x). Hence $DD\mathcal{M} = \mathcal{M}$.

The Cartan–Leray spectral sequence of homological type provides by dualization a spectral sequence of cohomological type as required, noticing that the functors D and L commute. □

Proof of Theorem 3.10. Let k be the trivial kG-module and let $P_. \to k \to 0$ be any kG-projective resolution of k. Consider the double complex whose (p, q) spot is $P_p \otimes_{kG} C_q(\mathcal{C}, L\mathcal{M})$ with the differentials given by the tensor product of those of the resolution of k and those of the Hochschild–Mitchell complex.

The homology of each row is usual group homology of G with coefficients in a free kG-module. Hence the homology of the q-row is concentrated in $p = 0$ where we have

$$H_0\big(G, C_q(\mathcal{C}, L\mathcal{M})\big) = C_q(\mathcal{C}, L\mathcal{M})/G = C_q(\mathcal{C}/G, \mathcal{M}).$$

The induced vertical differentials are the Hochschild–Mitchell boundaries ones by construction. Hence the filtration by the rows of the double complex provides at level 2 a $p = 0$ column of vector spaces which are $H_q(\mathcal{C}/G, \mathcal{M})$. The differentials at this level come and go to zero, consequently the abutment of the spectral sequence is the Hochschild–Mitchell homology of $\mathcal{B} = \mathcal{C}/G$ with coefficients in $\mathcal{M}$ as required.

The homology of the p-column is $P_p \otimes_{kG} H_.(\mathcal{C}, L\mathcal{M})$ with horizontal differentials given by the projective resolution of k. In other words the homology of the rows at level 1 is usual homology of G, namely $H_p(G, H_q(\mathcal{C}, L\mathcal{M}))$. □

Remark 3.12. An alternative approach for obtaining the Cartan–Leray spectral sequence of a Galois covering of k-categories is to use Grothendieck spectral sequence associated to the composition of two functors, see for instance [22, p. 150], in which case one have to check that the requirements of Grothendieck's Theorem are fulfilled either in the context of this paper or in the algebraic topology context. We thank Serge Bouc for pointing out this fact.

Remark 3.13. Let $\mathcal{C}$ be a small category which is not necessarily a k-category. The set theoretical nerve provides a simplicial set and the Cartan–Leray spectral sequence we have obtained is analogous to the spectral sequence of a fibration, see for instance [10, p. 157]. However we use in this paper k-categories obtaining simplicial vector spaces instead of simplicial sets. In other words the categories we consider have no multiplicative bases for the morphisms spaces in general, see for instance [2], a k-category is not in general the linearization of a category. The set theoretical approach cannot be used, moreover notice that Hochschild–Mitchell (co)homology has coefficients in a bimodule.

In case the group G is finite and the characteristic of the field is zero or do not divide the order of G, the algebra kG is semi-simple by Maschke's theorem. Consequently $H_p(G, X) = 0$ for $p > 0$ and for any kG-module X. So at level 2 the preceding spectral sequence has differentials zero, which shows the following.

Proposition 3.14. *Let $\mathcal{C} \to \mathcal{B}$ be a Galois covering of k-categories with finite group G and assume that the characteristic of the field k is zero or do not divide the order of G. Let $\mathcal{M}$ be a $\mathcal{B}$-bimodule. Then $H_n(\mathcal{B}, \mathcal{M}) = H_n(\mathcal{C}, L\mathcal{M})/G$.*

Moreover, if $\mathcal{M}$ is locally finite, then $H^n(\mathcal{B}, \mathcal{M}) = H^n(\mathcal{C}, L\mathcal{M})^G$.

Remark 3.15. Note that the change of the bimodule of coefficients is meaningful, namely the lifted bimodule provides the right context. This is related to a wrong interpretation of the present work made at the end of the introduction in [13], as the example below (first considered in [14]) illustrates.

Consider the free k-categories presented by the quivers

$$\mathcal{C}: \quad \begin{tikzcd} & x \arrow[dl, "a"'] \arrow[dr, "b"] & \\ y & & ty \\ & tx \arrow[ul, "tb"] \arrow[ur, "ta"'] & \end{tikzcd}$$

$$\mathcal{B}: \quad \bar{x} \underset{\bar{b}}{\overset{\bar{a}}{\rightrightarrows}} \bar{y}$$

where $\mathcal{B}$ is the Kronecker algebra. There is a Galois covering $\mathcal{C} \to \mathcal{B}$ under the action of the group $G = \langle t / t^2 = 1 \rangle$, $\dim H^1(\mathcal{C}, \mathcal{C}) = 1$ while $\dim H^1(\mathcal{B}, \mathcal{B}) = 3$. But in characteristic different from two $\dim H^1(\mathcal{C}, L\mathcal{B}) = 5$ and $\dim H^1(\mathcal{C}, L\mathcal{B})^G = 3$ as a simple computation shows.

We state another immediate consequence of the Cartan–Leray Theorem for Galois coverings of categories.

Proposition 3.16. *Let $\mathcal{C} \to \mathcal{B}$ be a Galois covering of k-categories with free group F and let $\mathcal{M}$ be a $\mathcal{B}$-bimodule. Then the vector spaces*

$$H_n(\mathcal{B}, \mathcal{M}) \quad \textit{and} \quad H_0\big(F, H_n(\mathcal{C}, L\mathcal{M})\big) \oplus H_1\big(F, H_{n-1}(\mathcal{C}, L\mathcal{M})\big)$$

are isomorphic.

Proof. Let F be a free group on a set B of generators. A free resolution of k is given by

$$0 \to \bigoplus_B kF \to kF \xrightarrow{\varepsilon} k \to 0.$$

For a kF-module X we have $H_i(kF, X) = 0$ for $i \geqslant 2$. Consequently the Cartan–Leray sequence has possibly non zero columns at level 2 only for $p = 0$ and $p = 1$, then all the differentials in this level are zero since they come from 0 or go to 0. □

Proposition 3.17. *With the same assumptions as before, assume in addition that $\mathcal{M}$ is locally finite. Then the vector spaces*

$$H^n(\mathcal{B}, \mathcal{M}) \quad \textit{and} \quad H^0\big(F, H^n(\mathcal{C}, L\mathcal{M})\big) \oplus H^1\big(F, H^{n-1}(\mathcal{C}, L\mathcal{M})\big)$$

are isomorphic.

Remark 3.18. If $F = T$ is free of rank one the resolution of k is

$$0 \to kT \xrightarrow{1-t} kT \xrightarrow{\varepsilon} k \to 0.$$

Consequently if X is a kT-module we have that the cohomology of X is the kernel and cokernel of the map given by $1 - t$, that is, $H^0(T, X) = X^T$ and $H^1(T, X) = X/T$. The preceding proposition specializes as follows for T an infinite cyclic group and $\mathcal{M}$ a locally finite bimodule:

$H^n(\mathcal{B}, \mathcal{M})$ and $H^n(\mathcal{C}, L\mathcal{M})^T \oplus H^{n-1}(\mathcal{C}, L\mathcal{M})/T$ are isomorphic vector spaces.

4. Rank of the Galois group and dimension of H^1

We recall first some well known definitions, see for instance [9]. A *basic* k-category has no different isomorphic objects. If $\mathcal{B}$ is not basic, choose one object in each isomorphism class and consider the full subcategory instead. This corresponds of course to Morita reduction of algebras and Hochschild–Mitchell (co)homology is invariant under this procedure, this can be proved by standard arguments.

A *split* k-category is a k-category $\mathcal{B}$ such that ${}_x\mathcal{B}_x$ is a local k-algebra for each object x. Notice that if a basic k-category is not split but has finite dimensional endomorphism algebras, the expansion process described in Section 2 provides a basic split category along the choice of a complete system of orthogonal primitive idempotents for each ${}_x\mathcal{B}_x$.

In case the field k is algebraically closed and B is a local finite dimensional k-algebra with maximal ideal M we have $B/M = k$ and there is a canonical decomposition $B = k \oplus M$. A k-category $\mathcal{B}$ is *totally split* if it is split and if for each object x we have such decomposition ${}_x\mathcal{B}_x = k \oplus M_x$. If k is algebraically closed every split category is totally split.

A k-category $\mathcal{B}$ is *hom-finite* if each vector space ${}_y\mathcal{B}_x$ is finite dimensional for each couple of objects (y, x) of $\mathcal{B}$, in other words the standard bimodule $\mathcal{B}$ is locally finite. Notice that we have already quoted that finite and hom-finite k-categories coincide with finite dimensional k-algebras provided with a complete system of orthogonal idempotents.

Finally a k-category is *connected* if the following equivalence relation $\sim$ has only one class: $\sim$ is generated by $x \sim y$ if and only if ${}_y\mathcal{C}_x \neq 0$ or ${}_x\mathcal{C}_y \neq 0$.

We will use the Cartan–Leray spectral sequence of cohomological type in order to show that for a connected Galois covering with group G of a totally split, basic and hom-finite k-category $\mathcal{B}$, the vector space of group homomorphisms $\operatorname{Hom}(G, k)$ is canonically embedded in $H^1(\mathcal{B}, \mathcal{B})$. As quoted in the introduction, Assem and de la Peña obtained this result in [1] when G is the fundamental group of a triangular finite dimensional algebra presented by a quiver with relations. In [19] de la Peña and Saorín noticed that the triangular hypothesis is superfluous.

We recall first a simple fact from the study of a first quadrant converging spectral sequence of cohomological type. Namely $E_{p+1}^{p,0}$ is a subspace of the abutment in degree p. Indeed the differentials at level $p+1$ at the spot $(p, 0)$ come from and go to 0, which shows that $E_{p+1}^{p,0} = E_{p+2}^{p,0} = \cdots = E_{\infty}^{p,0}$. In particular $E_2^{1,0}$ is a subspace of H^1.

Theorem 4.1. *Let $\mathcal{C} \to \mathcal{B}$ be a Galois covering of k-categories with group G where $\mathcal{C}$ is connected and $\mathcal{B}$ is hom-finite, basic and totally split. Then there is a canonical vector space monomorphism*

$$\operatorname{Hom}(G, k) \hookrightarrow H^1(\mathcal{B}, \mathcal{B}).$$

Proof. Since $\mathcal{B}$ is hom-finite we use the Cartan–Leray spectral sequence of cohomological type converging to $H^n(\mathcal{B}, \mathcal{B})$ which has level 2 term

$$E_2^{p,q} = H^p\big(G, H^q(\mathcal{C}, L\mathcal{B})\big).$$

At degree one we have a vector spaces exact sequence

$$0 \to E_2^{1,0} \to H^1(\mathcal{B}, \mathcal{B}) \to E_3^{0,1} \to 0$$

since $E_3^{0,1} = E_4^{0,1} = \cdots = E_\infty^{0,1}$.

Notice that $E_2^{1,0} = H^1(G, H^0(\mathcal{C}, L\mathcal{B}))$. We assert that the kG-module $H^0(\mathcal{C}, L\mathcal{B})$ has a one dimensional G-trivial direct summand. Recall that for any $\mathcal{C}$-bimodule $\mathcal{M}$ we have

$$H^0(\mathcal{C}, \mathcal{M}) = \big\{({}_xm_x)_{x \in \mathcal{C}} \mid {}_yc_x\, {}_xm_x = {}_ym_y\, {}_yc_x \text{ for all } {}_yc_x \in {}_y\mathcal{C}_x\big\}.$$

Moreover ${}_x(L\mathcal{B})_x = {}_{\bar{x}}\mathcal{B}_{\bar{x}} = k \oplus M_{\bar{x}}$. Consider a family

$$f = (\lambda_x + m_x)_{x \in \mathcal{C}_0} \in H^0(\mathcal{C}, L\mathcal{B}).$$

For $x \neq y$ and in case ${}_y\mathcal{C}_x$ or ${}_x\mathcal{C}_y$ is not zero, we have $\lambda_x = \lambda_y$: indeed, let $0 \neq c \in {}_y\mathcal{C}_x$. Since $\mathcal{B}$ is basic, $\bar{c}$ is not invertible and it belongs to the Jacobson radical r of the category $\mathcal{B}$ which is hom-finite. Then there exist a positive integer i such that $\bar{c} \in r^i$ but $\bar{c} \notin r^{i+1}$. Consider a vector space decomposition

$${}_{\bar{y}}\mathcal{B}_{\bar{x}} = k\bar{c} \oplus X \oplus \bar{y}r^{i+1}\bar{x}.$$

We have

$$\bar{c}(\lambda_x + m_x) = \bar{c}\lambda_x + \bar{c}m_x \quad \text{and} \quad (\lambda_y + m_y)\bar{c} = \lambda_y\bar{c} + m_y\bar{c}$$

and equality holds between these vectors. Notice that $\bar{c}m_x \in r^{i+1}$ and $m_y\bar{c} \in r^{i+1}$ while $\bar{c}\lambda_x \in k\bar{c}$ and $\lambda_y\bar{c} \in k\bar{c}$. Consequently $\bar{c}\lambda_x = \lambda_y\bar{c}$ and $\lambda_x = \lambda_y$.

Since $\mathcal{C}$ is connected we infer that $\lambda_x = \lambda_y$ for every pair of objects and $f = (\lambda + m_x)_{x \in \mathcal{C}_0}$. Clearly $(\lambda)_{x \in \mathcal{C}_0} \in H^0(\mathcal{C}, L\mathcal{B})$, consequently $(m_x)_{x \in \mathcal{C}_0} \in H^0(\mathcal{C}, L\mathcal{B})$. We have proved that

$$H^0(\mathcal{C}, L\mathcal{B}) = k \oplus \prod_{x \in \mathcal{C}_0} M_{\bar{x}} \cap H^0(\mathcal{C}, L\mathcal{B}).$$

Finally we note that this decomposition is a kG-modules decomposition since for each $s \in G$ the corresponding morphism changes the object spot of the vector space but remains the identity on the vector space itself. Moreover the resulting action on $k = (\lambda)_{x \in \mathcal{C}_0}$ is the identity.

We infer that $H^1(G, k)$ is a canonical direct summand of $E_2^{1,0}$, which in turn is canonically embedded in $H^1(\mathcal{B}, \mathcal{B})$.

Of course,

$$H^1(G, k) = \mathrm{Hom}\big(G, k^+\big)$$

since derivations with coefficients in a trivial module are just homomorphisms, and there are no inner derivations besides 0. □

The exact sequence at the beginning of the preceding proof provides a description of $H^1(\mathcal{B}, \mathcal{B})$ which can be explored further. In particular conditions for an isomorphism between $\mathrm{Hom}(G, k^+)$ and $H^1(\mathcal{B}, \mathcal{B})$ can be inferred, see [19].

The following result provides an upper bound for the rank of the Galois group of a covering, compare with [14] where the bound is obtained in case of a schurian category $\mathcal{C}$ and a free group acting on $\mathcal{C}$ with a finite dimensional basic and totally split algebra quotient.

Corollary 4.2. *Let $\mathcal{C} \to \mathcal{B}$ be a Galois covering of k-categories with group G where $\mathcal{C}$ is connected and $\mathcal{B}$ is basic, hom-finite, and totally split. Then* $\operatorname{rank} G \leqslant \dim H^1(\mathcal{B}, \mathcal{B})$.

Proof. By definition $\operatorname{rank} G = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} G^{\mathrm{ab}})$ where G^{ab} is the abelianization of G. We have

$$\dim_{\mathbb{Q}}\big(\mathbb{Q} \otimes_{\mathbb{Z}} G^{\mathrm{ab}}\big) = \dim_{\mathbb{Q}} \operatorname{Hom}\big(G, \mathbb{Q}^+\big) \leqslant \dim_k \operatorname{Hom}\big(G, k^+\big) \leqslant \dim_k H^1(\mathcal{B}, \mathcal{B}). \qquad \square$$

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